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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

COMPLEMENTING MAPS, CONTINUATION AND GLOBAL BIFURCATION

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ABSTRACT. We state, and indicate some of the consequences of, a theorem whose sole assumption is the nonvanishing of the Leray-Schauder degree of a compact vector field, and whose conclusions yield multidimensional existence, continuation and bifurcation results.

Complementing maps and the Theorem. Let X be a Banach space, m be a positive integer, and $O \subseteq \mathbf{R}^m \times X$ be open. Suppose $f: O \to X$ is an m-parameter compact vector field: i.e. $f(\lambda, x) = x - F(\lambda, x)$, for $(\lambda, x) \in O$, where F is continuous and maps bounded sets into relatively compact sets. A continuous map $g: O \to \mathbf{R}^m$, which maps bounded sets into bounded sets, will be called a *complement* for $f: O \to X$ provided that the Leray-Schauder degree, $\deg((g, f), O, 0)$, is defined and nonzero: $(g,f)((\lambda,x)) \equiv (g(\lambda,x), f(\lambda,x))$, for $(\lambda, x) \in O$, and since O is not assumed to be bounded, "defined" means $(g,f)^{-1}(0)$ is compact.

By cohomology we will mean Čech cohomology with integral coefficients. By dimension of a topological space we mean the Čech-Lebesgue covering dimension, and if $p \in A$, the space A will be said to have dimension at least m at p provided that each neighborhood, in A, of p has dimension at least m.

THEOREM. *Let X be a Banach space, m be a positive integer, and $O \subseteq \mathbf{R}^m \times X$ be open. Suppose that $f: O \to X$ is complemented by $g: O \to \mathbf{R}^m$. Then there exists a closed connected subset, C, of $f^{-1}(0)$, whose dimension at each point is at least m, and $(*)$ whenever K is a compact subset of C, with $g^{-1}(0) \cap C \subseteq K$, the map of pairs $g: (C, C-K) \to (\mathbf{R}^m, \mathbf{R}^m - 0)$ induces a nontrivial map in the mth cohomology group. In particular, $C \cap g^{-1}(0) \neq \varnothing$ and either C is unbounded or $\overline{C} \cap \partial O \neq \varnothing$. In the case when f and g are defined on $\overline{O}$ with $f^{-1}(0) \cap g^{-1}(0) \cap \partial O = \varnothing$, C also has the following properties: if C is bounded, then $\dim(\overline{C} \cap \partial O) \geq m-1$, when $m > 1$, and $\overline{C} \cap \partial O$ has at least two points, when $m = 1$; if $g: f^{-1}(0) \cap \overline{O} \to \mathbf{R}^m$ is proper and $\dim(\overline{C} \cap \partial O) < m-1$, then $g(\overline{C}) = \mathbf{R}^m$.*

SKELETON OF THE PROOF. Since $\deg((g,f), O, 0) \neq 0$, by using the cup-product in cohomology, it follows that whenever K is compact and $g^{-1}(0) \subseteq K \subseteq f^{-1}(0)$ the map $g: (f^{-1}(0), f^{-1} - K) \to (\mathbf{R}^m, \mathbf{R}^m - 0)$ is cohomologically nontrivial. Passing to the limit over all such K's we obtain a nontrivial class, ξ, in the mth Čech cohomology group with compact supports of $f^{-1}(0)$. The continuity of Čech theory enables us to choose a set, C, which is minimal

Received by the editors May 18, 1982.
1980 *Mathematics Subject Classification*. Primary 47H15, 35B32, 45G05.

among the closed subsets of $f^{-1}(0)$ to which ξ restricts nontrivially. This C has the properties claimed. $\square$

Some consequences of the Theorem. In what follows, $f\colon O \subseteq \mathbf{R}^m \times X \to X$ is an m-parameter compact vector field.

1. *Continuation under global hypotheses.* Let $\lambda_0 \in \mathbf{R}^m$ and let f_{λ_0} be the section of f over the slice O_{λ_0}. One shows that if $\deg(f_{\lambda_0}, O_{\lambda_0}, 0) \neq 0$ then $f\colon O \to X$ is complemented by $g\colon O \to \mathbf{R}^m$ defined by $g(\lambda, x) = \lambda - \lambda_0$. Thus the theorem furnishes a description of how $f^{-1}(0)$ emanates from O_{λ_0}. This is a multidimensional refinement of the Leray-Schauder continuation principle (see [**4**, **6** and **7**]).

2. *Continuation under local hypotheses.* Let $(\lambda_0, x_0) \in O$ and suppose that the map $x \to f(\lambda_0, x)$ has a Fréchet derivative, L, at $x = x_0$. Assume $L \in \mathcal{L}(X, X)$ is invertible. Then, letting $U = O - \{(\lambda_0, x) \mid x \neq x_0, f(\lambda_0, x) = 0\}$, one shows that $f\colon U \to X$ is complemented by $g\colon U \to \mathbf{R}^m$ defined by $g(\lambda, x) = \lambda - \lambda_0$. Thus, there is an m-dimensional connected subset, C, of $f^{-1}(0) \cap U$, which contains (λ_0, x_0), and which is either unbounded or $\overline{C} \cap \{\partial O \cup \{(\lambda_0, x) \mid x \neq x_0, f(\lambda_0, x) = 0\}\} \neq \varnothing$. Another global version of the implicit function theorem was obtained in [**3**].

3. *Nonlinear perturbation of linear Fredholm operators.* Let $\Omega \subseteq \mathbf{R}^2$ be simply connected, open and bounded, with $\partial\Omega$ a smooth closed curve. Suppose $\tau\colon \partial\Omega \to S^1$ is smooth and such that the winding number of $\tau\colon \partial\Omega \to S^1$ equals $-k < 0$. Given $\phi, \psi\colon \overline{\Omega} \times \mathbf{R}^2 \to \mathbf{R}$ we consider the following nonlinear Riemann-Hilbert problem: find $u, v\colon \overline{\Omega} \to \mathbf{R}$ such that, if $\tau = (\tau_1, \tau_2)$

$$\text{(R-H)} \quad \begin{array}{l} \text{(i)} \quad \begin{cases} u_x - v_y = \phi(x, y, u, v), \\ v_x + u_y = \psi(x, y, u, v) \end{cases} \text{in } \Omega, \\[2ex] \text{(ii)} \quad u\tau_1 - v\tau_2 = 0 \quad \text{on } \partial\Omega. \end{array}$$

Let $\alpha \in (0,1)$ be such that ψ and ϕ lie in $C^{1+\alpha}(\overline{\Omega} \times A, \mathbf{R})$ for each bounded subset A of $\mathbf{R}^2$ ($C^{1+\alpha}$ denotes the usual Schauder space). Under the assumption that $\psi(x, y, 0, 0) = \phi(x, y, 0, 0) = 0$, for each $(x, y) \in \overline{\Omega}$, it follows that for each $r > 0$, $\{(u, v) \in C^{1+\alpha}(\overline{\Omega}, \mathbf{R}^2) \mid (u, v) \text{ solves R-H}, \|(u, v)\|_{1+\alpha} = r\}$ has dimension at least $2k$.

Let $W = \{(u, v) \in C^{1+\alpha}(\overline{\Omega}, \mathbf{R}^2) | (u, v) \text{ satisfies (ii)}\}$, and let $L\colon W \to C^{\alpha}(\overline{\Omega}, \mathbf{R}^2)$ be the linear operator defined by the left-hand side of (i). Choose $z_1, \ldots, z_k$ in Ω and define $g\colon W \to \mathbf{R}^{2k+1}$ by

$$g((u, v)) = (u(z_1), v(z_1), \ldots, u(z_k), v(z_k), \int_{\partial\Omega} [\tau_1 v + \tau_2 u]\, ds).$$

Letting $X = g^{-1}(0)$, the linear theory (see [**10**]) implies $L\colon X \to C^{\alpha}(\overline{\Omega}, \mathbf{R}^2)$ has an inverse, T, and $W = V \oplus X$, with $\dim(V) = 2k + 1$.

If we rewrite (R-H) as $f((u, v)) \equiv T(L - H)((u, v)) = 0$, one shows that $f\colon V \oplus X \to X$ is complemented by g on each ball about the origin in W, and so we can apply the Theorem.

4. *Global bifurcation.* For simplicity, we assume $O = \mathbf{R}^m \times X$. We assume $\mathbf{R}^m \times \{0\} \subseteq f^{-1}(0)$, and call $\mathbf{R}^m \times \{0\}$ the trivial solutions of f. Suppose $\alpha, \beta \in \mathbf{R}^m$ are such that $(\alpha, 0)$ and $(\beta, 0)$ are not bifurcation points of $f^{-1}(0)$ and that

$\operatorname{ind}(f_\alpha, 0) \neq \operatorname{ind}(f_\beta, 0)$, where "ind" denotes the Leray-Schauder index. Then, if Γ is any open curve (i.e. homeomorphic image of $\mathbf{R}$) in $\mathbf{R}^m \times \{0\}$ which passes through $(\alpha, 0)$ and $(\beta, 0)$, there exists a connected set, C, of nontrivial zeros of f, whose dimension at each point is at least m, which intersects the segment, $\overline{(\alpha,0),(\beta,0)}$, of Γ, determined by $(\alpha, 0)$ and $(\beta, 0)$, and either C is unbounded or $\overline{C}$ intersects $\Gamma - \{\overline{(\alpha,0),(\beta,0)}\}$.

When $\alpha = 0$, $\beta = (1, 0, \ldots)$ and Γ is the line through α and β the proof runs as follows. Choose $r > 0$ such that $f(\lambda, x) \neq 0$ when $0 < \|x\| \leq r$ and either $|\lambda| \leq 3r$ or $|\lambda - \beta| \leq 3r$. Let $h\colon \mathbf{R} \to [0, r]$ be continuous, vanish outside of $[-r, 1+r]$, and equal r on $[r, 1-r]$. Then define $g\colon \mathbf{R}^m \times X \to \mathbf{R}^m$ by $g(\lambda_1, \ldots, \lambda_m) = (\|x\|^2 - (h(\lambda_1))^2, \lambda_2, \ldots, \lambda_m)$.

One shows that if $U = \mathbf{R}^m \times \{X - \{0\}\}$, then $\deg((g, f), U, 0) = \operatorname{ind}(f_\beta, 0) - \operatorname{ind}(f_\alpha, 0)$, and so g complements f on U. So we extract the subset, C, of $f^{-1}(0) \cap U$, having the properties in the conclusion of the Theorem. Conclusion $(*)$ implies our assertions regarding $\overline{C} \cap \Gamma$.

This bifurcation result yields the principle abstract global bifurcation results of [**9** and **1**]. J. Ize (see [**8**]) has given a proof of the bifurcation theorem in [**9**] using a map similar to the above g.

REMARK. In the definition of complementing map if one replaces the Leray-Schauder degree by the Browder-Petryshyn degree for A-proper mappings (see [**5**]) the Theorem still holds. We believe that approximation results similar to those used in [**2**] will also yield the Theorem when F is assumed to be condensing.

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